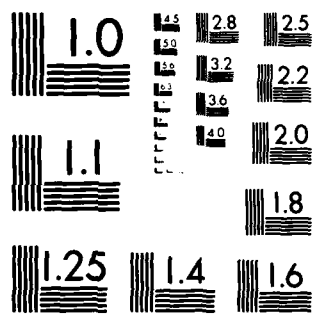


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FINAL REPORT FOR  
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THE ANALYTIC STRUCTURE OF ORDINARY AND PARTIAL  
DIFFERENTIAL EQUATION

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## THE ANALYTIC STRUCTURE OF ORDINARY AND PARTIAL DIFFERENTIAL EQUATION

This report is on work supported by the AFOSR in the period from March 1983 to March 1984. During this time our principal efforts were the development of methods for solving nonlinear ordinary and partial differential equations. Specifically, we now find it possible to identify when an equation is integrable, and, when integrable, to "find" the solution. Previous to March 1983 we defined a "Painlevé property" for partial differential equations [1] and a new type of Bäcklund transformation. This Bäcklund transformation leads naturally to equations formulated in terms of the Schwarzian derivative [2]. Since the Schwarzian derivative is (classically) related to a projective representation of "linear" equations [3], this suggested a method for "linearizing" the nonlinear equation (in effect, to find the Lax Pair). Initially the method was based on a Wronskian technique [2] that required the assumption (guess) of the linear equations and a subsequent verification of their consistency. The nonconstructive element of this approach made applications to sufficiently complicated equations difficult. Therefore a primary task at the beginning of the grant period was the development of a constructive method for finding the Lax Pair of equations with the Painlevé property. Another task was to determine the "integrable reductions" of nonintegrable systems.

In Ref. [4] we have found conditions for the integrable reductions of the double and  $(N+1)$  Sine-Gordon equations. Also in this reference we found, by an analysis of the Hirota-Satsuma equations, a direct method for finding the Lax-Pair. That is, from the Bäcklund transformation, equations are found that are formulated in terms of the Schwarzian derivative. With these "modified" equations there is found a Miura transformation (Riccati-type equation) which linearizes into the Lax Pair for the original system. Thus, the construction of the Lax Pair is reduced to calculation. Furthermore, from the "modified" equations and

their Bäcklund transformations (symmetries), rational solutions can be found [4, 5, 6]. The Miura transformation can also be used to define the recursion operators for the sequence of "higher order" equations [4, 6].

In reference [5] certain sequences of higher order equations are examined. Using the recursion operator, Bäcklund transformations are iteratively defined for the equations in the sequence. From the symmetries of the "modified" equations (1) formula are derived for the iterative construction of rational solutions, and (2) the equations in the sequence are shown to possess the Painlevé property.

In reference [6] the procedure of Bäcklund transformation/Modified Equations is formulated and applied to the sequence of Bousinesq equations. Using the discrete symmetries ( Bäcklund transformations) of the "modified" sequence (1) it is shown that the Bousinesq/Modified Bousinesq sequence has the Painlevé property, and (2) the rational solutions are defined iteratively. For the nonlinear Schrodinger equation a scalar Lax Pair is discovered by the above method. The "modified" NLS equations are found to have a degenerate form of Bäcklund transformation (a reduction to Burgers' Eq.) which precludes the construction of nontrivial rational solutions. This, in turn, suggests a criterion for the existence of rational solutions.

In reference [7] we begin the study of ordinary differential equations by the Bäcklund method. The possibility of recursively linearizing nonlinear odes by this method is explored. Also the "canonical" classification of integrable odes as occurrences of "Novikov" equations is currently being examined. "Novikov" equations are the steady state equations of sequences of integrable pdes. And there exists powerful methods for algorithmically discovering their integrals.

The publications produced with AFOSR support during this contract period are refs. [4, 5, 6, 7]. The title pages and abstracts are contained in the Appendix.

# REFERENCES

1. John Weiss, M. Tabor and George Carnevale, "The Painlevé Property for partial differential equations," J. Math. Phys. **24**, 522 (1983).
2. John Weiss, "The Painlevé property for partial differential equations. II: Bäcklund transformation, Lax Pair and the Schwarzian derivative," J. Math. Phys. **24**, 1405 (1983).
3. Einar Hille, "Ordinary Differential Equations in the Complex Domain," John Wiley and Sons, Inc., New York, 1976.
4. John Weiss, "The Sine-Gordon Equations: Complete and Partial Integrability," J. Math. Phys. **25**, July (1984).
5. John Weiss, "On Classes of Integrable Systems and the Painlevé property," J. Math. Phys., **25**, 13 (1984).
6. John Weiss, "The Painlevé property and Bäcklund transformations for the sequence of Bousinesq equations," sub. J. Math. Phys. (1984).
7. John Weiss, " Bäcklund transformation and linearizations of the Henon-Heiles systems," Phys. Letts A **102A**, 329 (1984).

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**APPENDIX:**

Publications with AFOSR support during contract period, March 1983 to  
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**THE SINE-GORDON EQUATIONS;  
COMPLETE AND PARTIAL INTEGRABILITY**

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### ABSTRACT

The Sine-Gordon equation in one space- one time dimension is known to possess the Painlevé property and to be completely integrable. It is shown how the method of "singular manifold" analysis obtains the Bäcklund transform and the Lax Pair for this equation. A connection with the sequence of higher order KdV equations is found. The "modified" Sine-Gordon equations are defined in terms of the singular manifold. These equations are shown to be identically Painlevé. Also, certain "rational" solutions are constructed iteratively.

The double Sine-Gordon equation is shown not to possess the Painlevé property. However, if the singular manifold defines an "affine minimal surface," then the equation has integrable solutions. This restriction is termed "partial integrability."

The Sine-Gordon equation in  $(N+1)$  variables ( $N$  space, 1 time) where  $N$  is greater than one is shown not to possess the Painlevé property. The condition of partial integrability requires the singular manifold to be an "Einstein space with null scalar curvature." The known integrable solutions satisfy this constraint in a trivial manner.

Finally, the coupled KdV, or Hirota-Satsuma, equations possess the Painlevé property. The associated "modified" equations are derived and from these the Lax Pair is found.

# On classes of integrable systems and the Painlevé property<sup>a)</sup>

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The Caudrey–Dodd–Gibbon equation is found to possess the Painlevé property. Investigation of the Bäcklund transformations for this equation obtains the Kuperschmidt equation. A certain transformation between the Kuperschmidt and Caudrey–Dodd–Gibbon equation is obtained. This transformation is employed to define a class of p.d.e.'s that identically possesses the Painlevé property. For equations within this class Bäcklund transformations and rational solutions are investigated. In particular, the sequences of higher order KdV, Caudrey–Dodd–Gibbon, and Kuperschmidt equations are shown to possess the Painlevé property.

PACS numbers: 02.30. + g

## 1. INTRODUCTION

In Ref. 1 the Painlevé property for partial differential equations was defined. Briefly, we say that a partial differential equation has the Painlevé property when the solutions of the p.d.e. are "single-valued" about the movable, singularity manifold and the singularity manifold is "noncharacteristic." To be precise, if the singularity manifold is determined by

$$\varphi(z_1, z_2, \dots, z_n) = 0 \quad (1.1)$$

and  $u = u(z_1, \dots, z_n)$  is a solution of the p.d.e., then we require that

$$u = \varphi^\alpha \sum_{j=0}^{\infty} u_j \varphi^j, \quad (1.2)$$

where  $u_0 \neq 0$ ,  $\varphi = \varphi(z_1, \dots, z_n)$ ,  $u_j = u_j(z_1, \dots, z_n)$  are analytic functions of  $(z_j)$  in a neighborhood of the manifold (1.1), and  $\alpha$  is an integer. The requirement that the manifold (1.1) be noncharacteristic insures that the expansion (1.2) will be well defined, in the sense of the Cauchy–Kowalevsky theorem. Substitution of (1.2) into the p.d.e. determines the value(s) of  $\alpha$ , and defines the recursion relations for  $u_j$ ,  $j = 0, 1, 2, \dots$ . When the ansatz (1.2) is correct, the p.d.e. is said to possess the Painlevé property and is conjectured to be integrable. The "Painlevé conjecture," as originally formulated by Ablowitz *et al.*,<sup>2</sup> states that when all the ordinary differential equations obtained by exact similarity transforms from a given partial differential equation have the Painlevé property, then the partial differential equation is "integrable." The above definition of the "Painlevé property" allows this conjecture to be stated directly for the partial differential equation.

In Ref. 3 Bäcklund transformations were obtained by truncating the expansion (1.2) at the "constant" level term. That is, we set

$$u = u_0 \varphi^{-N} + u_1 \varphi^{-N+1} + \dots + u_N \quad (1.3)$$

and find, from the recursion relations for  $u_j$ , an overdeter-

mined system of equations for  $(\varphi, u_j, j = 0, 1, \dots, N)$ , where  $u_N$  will satisfy the (original) p.d.e. Upon solving the overdetermined system, it was found, for those equations considered, that  $\varphi$  satisfied an equation formulated in terms of the Schwarzian derivative:

$$\{\varphi; x\} = \frac{\partial}{\partial x} \left( \frac{\varphi_{xx}}{\varphi_x} \right) - \frac{1}{2} \left( \frac{\varphi_{xx}}{\varphi_x} \right)^2. \quad (1.4)$$

The invariance of (1.4) under the Moebius group

$$\varphi = \frac{a\psi + b}{c\psi + d}, \quad \{\varphi; x\} = \{\psi; x\} \quad (1.5)$$

motivates the substitution

$$\varphi = v_1/v_2, \quad (1.6)$$

by which the Lax pairs may be found.<sup>3</sup>

Investigation of a certain class of equations formulated in terms of the Schwarzian derivatives revealed that these equations have the Painlevé property about movable, singularity manifolds of order  $-1$ . However, the occurrence of an additional type of movable singularity prevents this class of equations from identically possessing the Painlevé property. Hence, nonintegrable behavior can arise.<sup>2</sup>

In this paper a restriction (symmetry) is imposed that allows one to conclude that, when an equation is formulated in terms of the Schwarzian derivative and has this "symmetry," the equation identically possesses the Painlevé property. Within this class of equations are found the KdV, Caudrey–Dodd–Gibbon and Kuperschmidt equations. Furthermore, the "symmetry" property and invariance under the Moebius group allow effective Bäcklund transforms to be defined for these equations. In particular, rational or algebraic  $[\ln(x, t)]$  solutions can be generated iteratively.

In the next section, the Painlevé property and Bäcklund transformation for the KdV equation are reviewed for later reference.

In Sec. 3 the Painlevé property and Bäcklund transforms for the Caudrey–Dodd–Gibbon equation are presented. From these considerations the Kuperschmidt equation is found. The transformation between the Caudrey–Dodd–Gibbon and Kuperschmidt equations can be regarded as a

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**THE PAINLEVE PROPERTY AND BACKLUND TRANSFORMATIONS  
FOR THE SEQUENCE OF BOUSINESQ EQUATIONS**

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## ABSTRACT

We investigate the sequence of Bousinesq equations by the method of singular manifolds. For the Bousinesq Equation, which is known to possess the Painlevé property, a Bäcklund transformation is defined. This Bäcklund transformation, which is formulated in terms of the Schwarzian derivative, obtains the system of modified Bousinesq equations and the resulting Miura-type transformation. The modified Bousinesq equations are found to be invariant under a discrete group of symmetries, acting on the dependent variables. By linearizing the Miura transformation (and modified equations) the Lax Pair is readily obtained.

Furthermore, by a result of Fokas and Anderson, the recursion operators defining the sequence of (higher order) Bousinesq equations may be constructed from the Miura transformation. This allows the (recursive) definition of Bäcklund transformations for this sequence of equations. The recursion operator is found to preserve the discrete symmetries of the modified Bousinesq equations. This leads to the conclusion that the sequences of Bousinesq and modified Bousinesq equations identically possess the Painlevé property (are meromorphic). We also find that, by a simple reduction, the sequences of Caudrey-Dodd-Gibbon and Kuperschmidt equations are contained within the Bousinesq sequence.

Rational solutions are iteratively constructed for the Bousinesq equation and a criterion is proposed for the existence of rational solutions of general integrable systems.

## BÄCKLUND TRANSFORMATION AND LINEARIZATIONS OF THE HENON-HEILES SYSTEM\*

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When the Hénon-Heiles system possesses the Painlevé property certain Bäcklund transformations defined in terms of the manifold of singularities. The resulting system of equations are shown to effectively linearize the Hénon-Heiles system.

A system of nonlinear ordinary differential equations is said to possess the Painlevé property when all the "movable" singularities are simple poles [1]. Since the work of Kovalevskaya on the motion of a rigid body about a fixed point, the Painlevé property, in its various guises, has been proposed as a criterion for complete integrability [2]. Recently, it is found that: (1) the Painlevé property is a necessary condition for "algebraically complete integrability" in terms of "abelian functions" [3]. (2) If a system misses the Painlevé property by a certain degree (has complex or irrational "resonances"), the system cannot be algebraically integrable [4].

Now, in ref. [5], we showed that the Painlevé property, when formulated in terms of a "singular manifold", has a natural extension to systems of nonlinear partial differential equations. If the manifold of singularities is determined by

$$\varphi(z_1, z_2, \dots, z_n) = 0 \quad (1)$$

and  $u = u(z_1, \dots, z_n)$  is a solution of the p.d.e., then it is required that:

$$u = \varphi^{\alpha} \sum_{j=0}^{\infty} u_j \varphi^j, \quad (2)$$

\* Work supported by Department of Energy Contract DOE DE-AC03-81ER10923 and U.S. Air Force Grant No. AFOSR 83-0095.

where  $u_0 \neq 0$ ,  $\varphi = \varphi(z_1, \dots, z_n)$ ,  $f = f(z_1, \dots, z_n)$  are analytic functions of  $z_j$  in a neighborhood of (1), and  $\alpha$  is a (negative) integer. Also, the manifold (1) is assumed to be "noncharacteristic" so that the "single-valued" expansion (2) about the "movable" singularity (1) will be well defined, in the sense of the Cauchy-Kovalevskaya theorem. When (2) is correct the p.d.e. is said to possess the Painlevé property, and is conjectured to be integrable.

The classical definition of the Painlevé property (for o.d.e.'s) is obtained when

$$\varphi(t) = t - t_0. \quad (3)$$

However, it is still possible (for o.d.e.'s) to allow  $\varphi = \varphi(t)$  to be an arbitrary function (i.e., expand about the "zeros" of  $\varphi$ ) as long as  $\varphi_t \neq 0$  near (1)  $\varphi$  is "noncharacteristic". Herein, we find that this allows Bäcklund transformation to be defined for ordinary differential equations.

In general, Bäcklund transformations are obtained by "truncating" (2) at the "constant" level term. That is, we set:

$$u = u_0 \varphi^{-N} + u_1 \varphi^{-N+1} + \dots + u_n, \quad (4)$$

and find, from the recursion relations for  $u_j$ , an overdetermined system of equations for  $\varphi$ ;  $u_j$ ;  $j = 0, 1, \dots, N$ , when  $u_n$  will satisfy the (original) equation [6-9]. In ref. [7] Bäcklund transformations were defined,

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THE ANALYTIC STRUCTURE OF ORDINARY AND PARTIAL DIFFERENTIAL EQUATION

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During this time principal efforts were the development of methods for solving nonlinear ordinary and partial differential equations. Specifically, we now find it possible to identify when an equation is integrable, and, when integrable, to "find" the solution. Previous to March 1983 we defined a "Painleve property" for partial differential equations and a new type of Backlund transformation. This Backlund transformation leads naturally to equations formulated in terms of the Schwarzian derivative. Since the Schwarzian derivative is (classically) related to a projective representation of "linear" equations, this suggested a method for "linearizing" the nonlinear equation (in effect, to find the Lax Pair). Initially the method was based on a Wronskian technique that required the assumption (guess) of the linear equations and a subsequent verification of their consistency. The nonconstructive element of this approach made applications to sufficiently complicated equations difficult. Therefore a primary talk at the beginning of the grant period was the development of a constructive method for finding the Lax Pair of (CONTINUED)

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